Minimum-Energy Guidance Law for Spacecraft Rendezvous in the Hill Frame

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ME 8220 – Continuous Time Optimal Control

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Rendezvous and Proximity Operations (RPO)

RPO is fundamental to modern space operations

- Resupplying ISS
- On-orbit satellite servicing
- Future deep space assembly missions

Reducing fuel use during RPO is crucial since fuel is limited once on-orbit, driving the need for optimal/energy efficient paths.

200+
ISS resupply missions completed since 1998

$300M+
Cost of single ISS resupply mission

3D Minimum-Energy Rendezvous Problem

Objectives:
1. Drive from initial relative position/velocity
2. Achieve desired docking state in Hill frame
3. Minimize control effort throughout maneuver

Assumptions:
- Circular orbit - LTI system (A and B are const.)
- Deputy/Chief are point masses with small separation
- Less $\boldsymbol{u}(t)$ means less fuel consumption
- Idealized actuator model (continuous acc.)
- Assume full state is known perfectly (i.e. no disturbances - J2, SRP, sensor noise, etc.)
- No collision avoidance, keep-out zones, approach corridor constraints

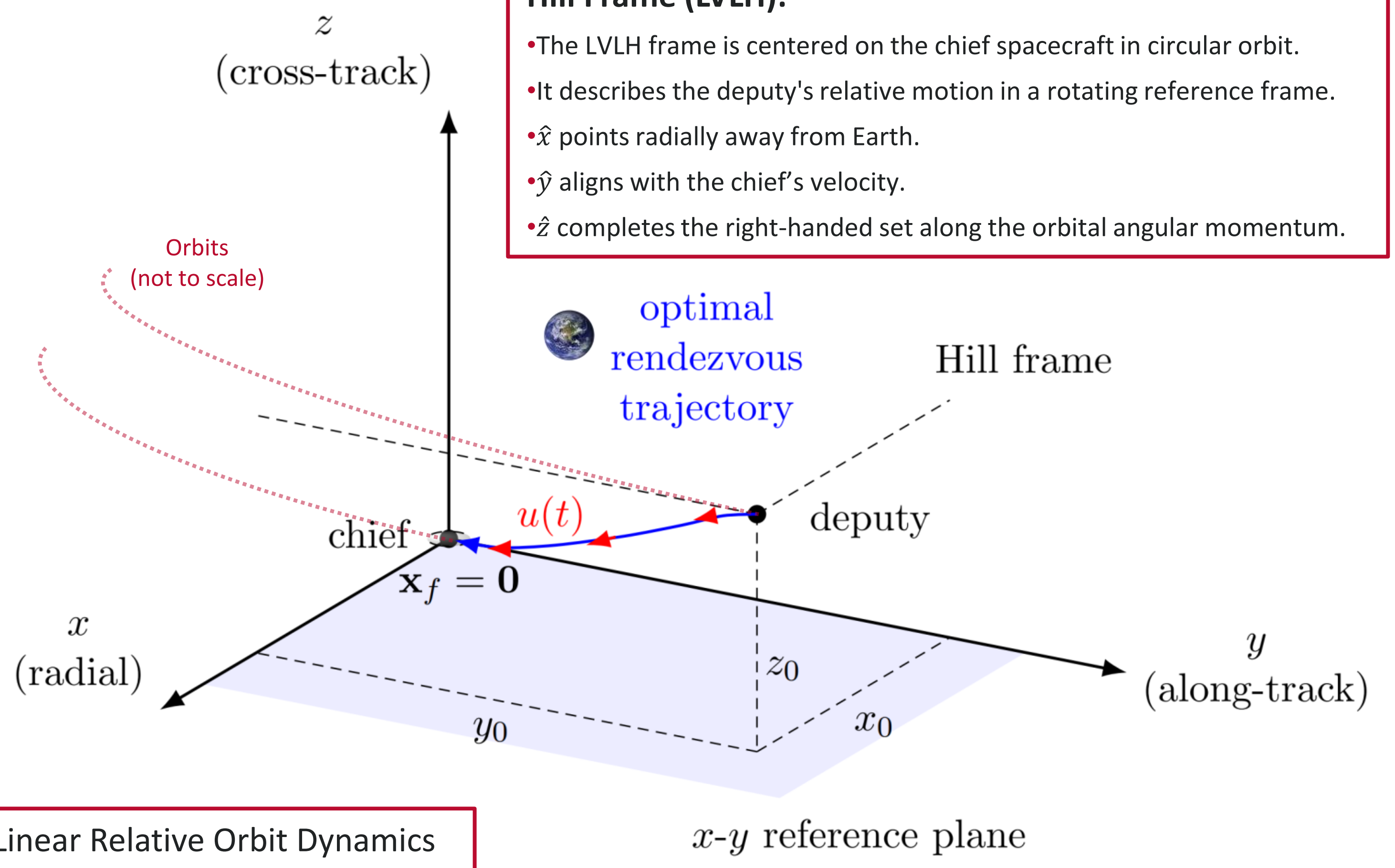
$$\mathbf{x}(t) = \begin{bmatrix} x(t) & y(t) & z(t) & \dot{x}(t) & \dot{y}(t) & \dot{z}(t) \end{bmatrix}^\top,$$
$$\mathbf{u}(t) = \begin{bmatrix} u_x(t) & u_y(t) & u_z(t) \end{bmatrix}^\top$$

$$\min_{\mathbf{u}(t)} \quad J = \frac{1}{2}\int_0^{t_f} \mathbf{u}^\top(t)\, R\, \mathbf{u}(t)\, dt$$

$$\text{subject to} \quad \ddot{x} - 2n\dot{y} - 3n^2 x = u_x,$$
$$\ddot{y} + 2n\dot{x} = u_y,$$
$$\ddot{z} + n^2 z = u_z,$$
$$\mathbf{x}(0) = \mathbf{x}_0, \qquad \mathbf{x}(t_f) = \mathbf{x}_f = \mathbf{0}.$$

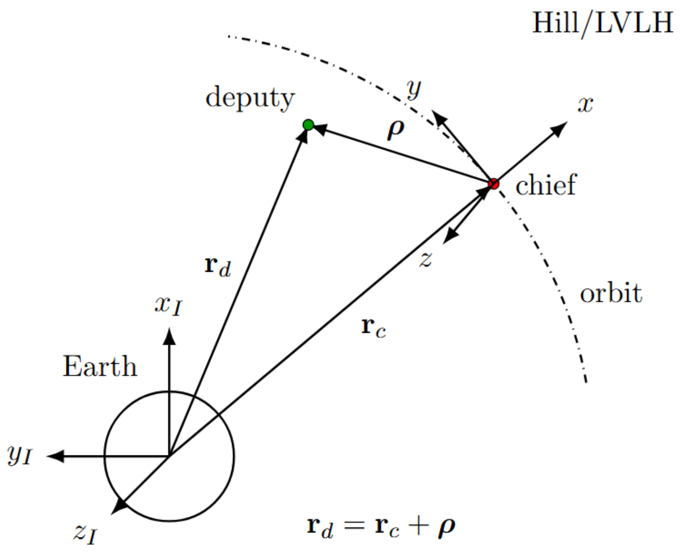
3D Linear Relative Orbit Dynamics (Clohessy-Wiltshire/CW Equations)

Hill Frame (LVLH):
- The LVLH frame is centered on the chief spacecraft in circular orbit.
- It describes the deputy's relative motion in a rotating reference frame.
- $\hat{x}$ points radially away from Earth.
- $\hat{y}$ aligns with the chief's velocity.
- $\hat{z}$ completes the right-handed set along the orbital angular momentum.



Linear State-Space Clohessy-Wiltshire Relative Orbit Equations of Motion

Relative Orbit Geometry (deputy and chief)



$\mathbf{r}_d$: deputy position vector in the inertial frame

$\mathbf{r}_c$: chief position vector in the inertial frame

$\boldsymbol{\rho}$: relative position vector from chief to deputy

$\mathbf{u}$: control acceleration vector applied to the deputy

Nonlinear Relative Orbit Dynamics

$$\mathbf{r}_d = \mathbf{r}_c + \boldsymbol{\rho}, \qquad \boldsymbol{\rho} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\ddot{\mathbf{r}}_d = \ddot{\mathbf{r}}_c + \left(\frac{d^2\boldsymbol{\rho}}{dt^2}\right)_I$$

$$\left(\frac{d^2\boldsymbol{\rho}}{dt^2}\right)_I = \ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

$$\ddot{\mathbf{r}}_c = -\mu \frac{\mathbf{r}_c}{\|\mathbf{r}_c\|^3}, \qquad \ddot{\mathbf{r}}_d = -\mu \frac{\mathbf{r}_d}{\|\mathbf{r}_d\|^3} + \mathbf{u}$$

$$\boxed{\begin{aligned} &\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \\ &= -\mu \left(\frac{\mathbf{r}_c + \boldsymbol{\rho}}{\|\mathbf{r}_c + \boldsymbol{\rho}\|^3} - \frac{\mathbf{r}_c}{\|\mathbf{r}_c\|^3} \right) + \mathbf{u} \end{aligned}}$$

$\boldsymbol{\omega}$: angular velocity of the Hill/LVLH frame relative to inertial

$\dot{\boldsymbol{\omega}}$: angular acceleration of the Hill/LVLH frame

μ : gravitational parameter of the central body

n : mean motion of the chief circular orbit

r_0 : chief orbit radius for the circular reference orbit

The Clohessy-Wiltshire (CW) Equations

Assume: $\|\boldsymbol{\rho}\| \ll r_0, \quad \boldsymbol{\omega} = n\hat{\mathbf{z}}, \quad \dot{\boldsymbol{\omega}} = 0, \quad n^2 = \frac{\mu}{r_0^3}$

$$\boxed{\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= u_x, \\ \ddot{y} + 2n\dot{x} &= u_y, \\ \ddot{z} + n^2 z &= u_z \end{aligned}}$$

$$\mathbf{x}(t) = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^\top, \qquad \mathbf{u}(t) = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^\top$$

$$\boxed{\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

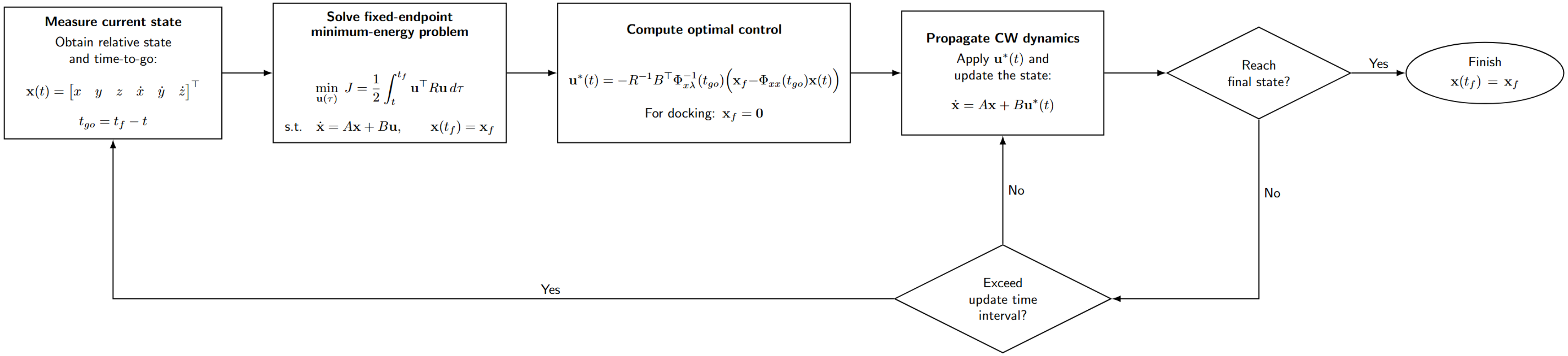
Gravity-gradient

Coriolis coupling (due to rotational Hill frame)

Cross-track restoring (natural)

Optimal Rendezvous Guidance Law

Formulate a fixed-endpoint minimum-energy rendezvous problem using the CW dynamics.



CW dynamics

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$\Longrightarrow$

Minimum-energy cost

$$J = \frac{1}{2}\int_t^{t_f} \mathbf{u}^\top R \mathbf{u}\, d\tau$$

$\Longrightarrow$

Hamiltonian & optimality

$$H = \frac{1}{2}\mathbf{u}^\top R \mathbf{u} + \boldsymbol{\lambda}^\top (A\mathbf{x} + B\mathbf{u})$$

$$\frac{\partial H}{\partial \mathbf{u}} = 0 \Rightarrow \mathbf{u}^*(t) = -R^{-1}B^\top \boldsymbol{\lambda}(t)$$

$\Longrightarrow$

Costate/STM relation

$$\dot{\boldsymbol{\lambda}} = -A^\top \boldsymbol{\lambda}$$

$$\mathbf{x}_f = \Phi_{xx}(t_{go})\mathbf{x}(t) + \Phi_{x\lambda}(t_{go})\boldsymbol{\lambda}(t)$$

$\Longrightarrow$

Optimal rendezvous guidance law

$$\boldsymbol{\lambda}(t) = \Phi_{x\lambda}^{-1}(t_{go})\Big(\mathbf{x}_f - \Phi_{xx}(t_{go})\mathbf{x}(t)\Big)$$

$$\boxed{\mathbf{u}^*(t) = -R^{-1}B^\top \Phi_{x\lambda}^{-1}(t_{go})\Big(\mathbf{x}_f - \Phi_{xx}(t_{go})\mathbf{x}(t)\Big)}$$

Docking: $\mathbf{x}_f = \mathbf{0}, \qquad t_{go} = t_f - t$

$$\Phi(t_{go}) = e^{F t_{go}}, \qquad F = \begin{bmatrix} A & -BR^{-1}B^\top \\ 0 & -A^\top \end{bmatrix}, \qquad \Phi(t_{go}) = \begin{bmatrix} \Phi_{xx}(t_{go}) & \Phi_{x\lambda}(t_{go}) \\ \Phi_{\lambda x}(t_{go}) & \Phi_{\lambda\lambda}(t_{go}) \end{bmatrix}$$

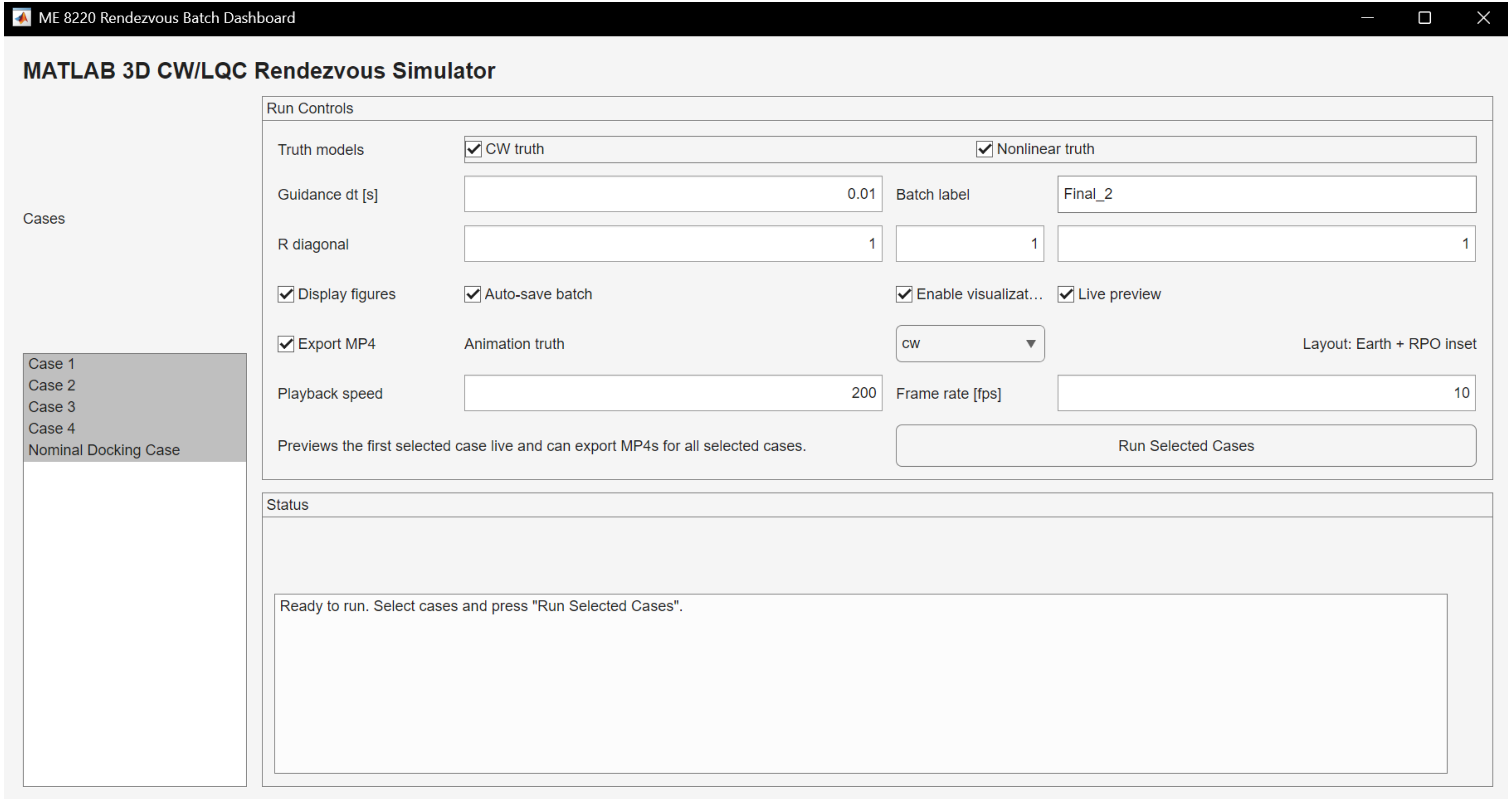
Notation: $\mathbf{x}(t)$ = relative state, $\mathbf{u}(t)$ = control acceleration, A, B = CW system matrices, R = control-weighting matrix, $\boldsymbol{\lambda}(t)$ = costate, $\Phi_{xx}, \Phi_{x\lambda}$ = augmented STM blocks, $\mathbf{x}_f$ = terminal state, $t_{go} = t_f - t$ = time-to-go.

Simulation: Satellite Proximity Operation Scenarios

- Guidance law implemented into MATLAB
- 4 test cases of deputy rendezvous with a chief satellite given initial and final conditions
- Also nominal dock case (ISS)

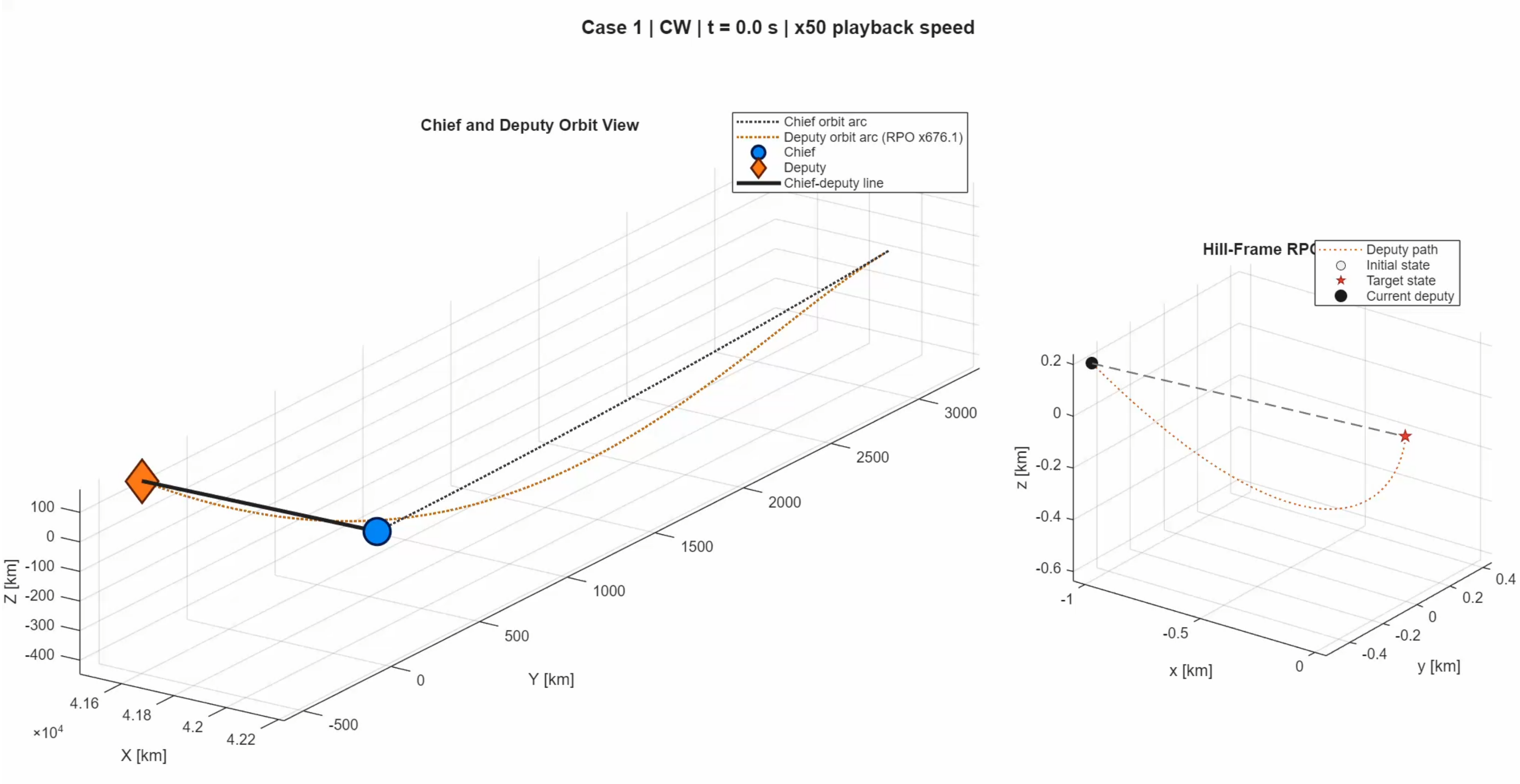
Parameter	Case 1	Case 2	Case 3	Case 4	Nominal Dock
x_0 (km)	-1.0	-1.0	-0.5	-0.5	-0.1
y_0 (km)	-0.5	1.0	0.1	0.1	-0.05
z_0 (km)	0.2	0.0	0.0	0.0	0.02
$v_{x,0}$ (m/s)	0.0	5.0	5.0	5.0	0.0
$v_{y,0}$ (m/s)	5.0	0.0	-5.0	-5.0	0.2
$v_{z,0}$ (m/s)	-5.0	10.0	0.0	0.0	-0.1
$r_{T,x}$ (km)	42169.0	42169.0	42169.0	42169.0	42169.0
$r_{T,y}$ (km)	0.0	0.0	0.0	0.0	0.0
$v_{T,x}$ (m/s)	0.0	0.0	0.0	0.0	0.0
$v_{T,y}$ (m/s)	3074.7	3074.7	3074.7	3074.7	3074.7
t_f (s)	1000	1000	1000	1500	600
x_f (km)	0.0	-1.0	0.0	-10.0	0.0
y_f (km)	0.0	0.0	0.0	0.0	0.0
z_f (km)	0.0	0.0	0.0	0.0	0.0
$v_{x,f}$ (m/s)	0.0	0.1	0.0	1.0	0.0
$v_{y,f}$ (m/s)	0.0	0.0	0.0	0.0	0.0
$v_{z,f}$ (m/s)	0.0	0.0	0.0	0.0	0.0

MATLAB Dashboard for Batch Runs



Case 1: Baseline 3D Docking Rendezvous

- Rendezvous from relative offset with both out-of-plane displacement and nonzero cross-track/closing velocity
- Full rendezvous at origin with zero terminal velocity
- Completed over 1000 s with zero terminal error
- Peak control: 0.0258 m/s^2
- Control cost (integrated control effort): 0.0853 m^2/s^3



Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 1	-1.0	-0.5	0.2	0.0	5.0	-5.0	42169.0	0.0	0.0	3074.7	1000	0.0	0.0	0.0	0.0	0.0	0.0

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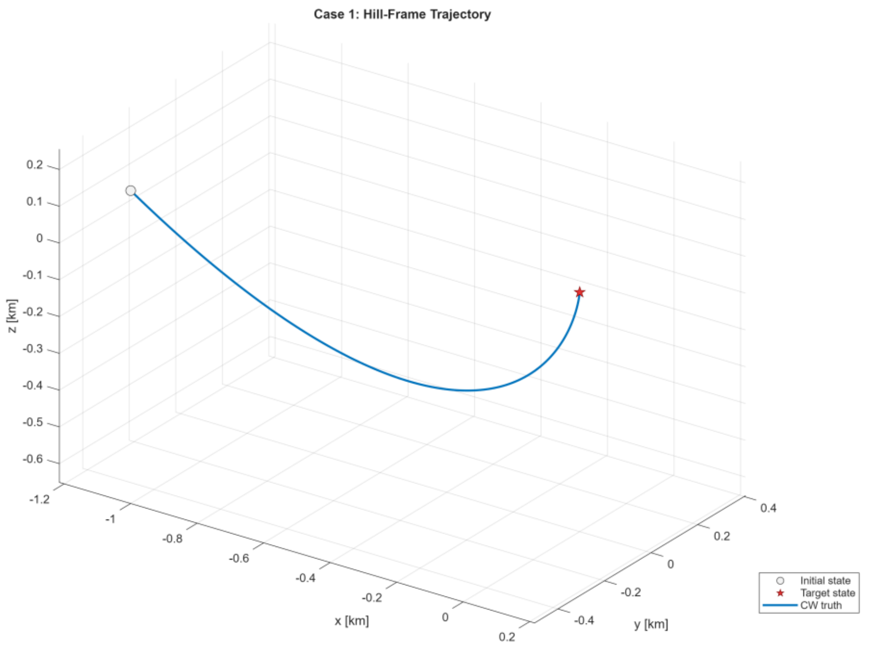

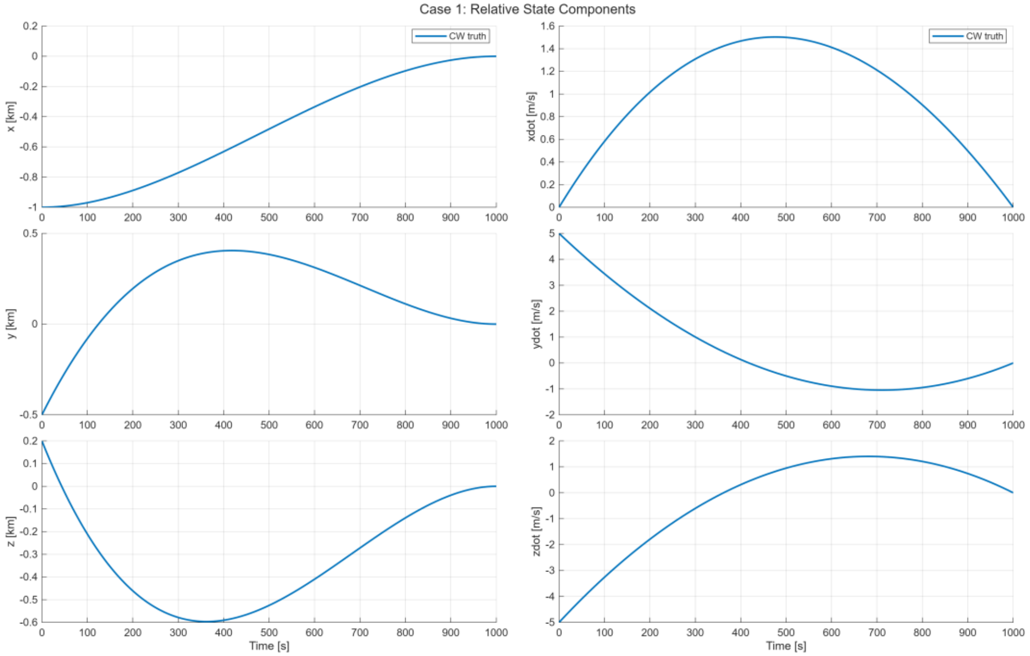

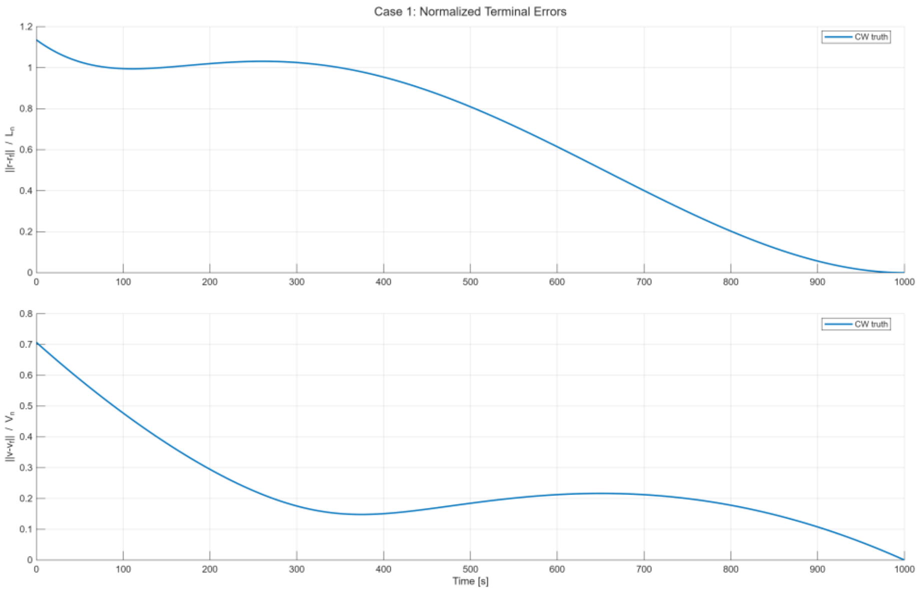

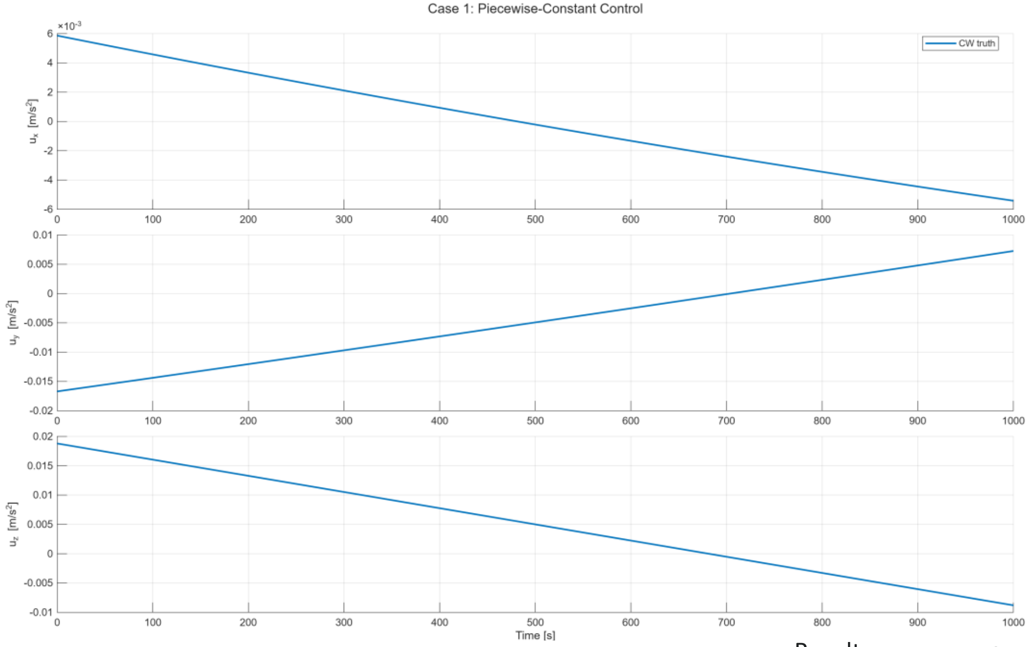

Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 1	-1.0	-0.5	0.2	0.0	5.0	-5.0	42169.0	0.0	0.0	3074.7	1000	0.0	0.0	0.0	0.0	0.0	0.0

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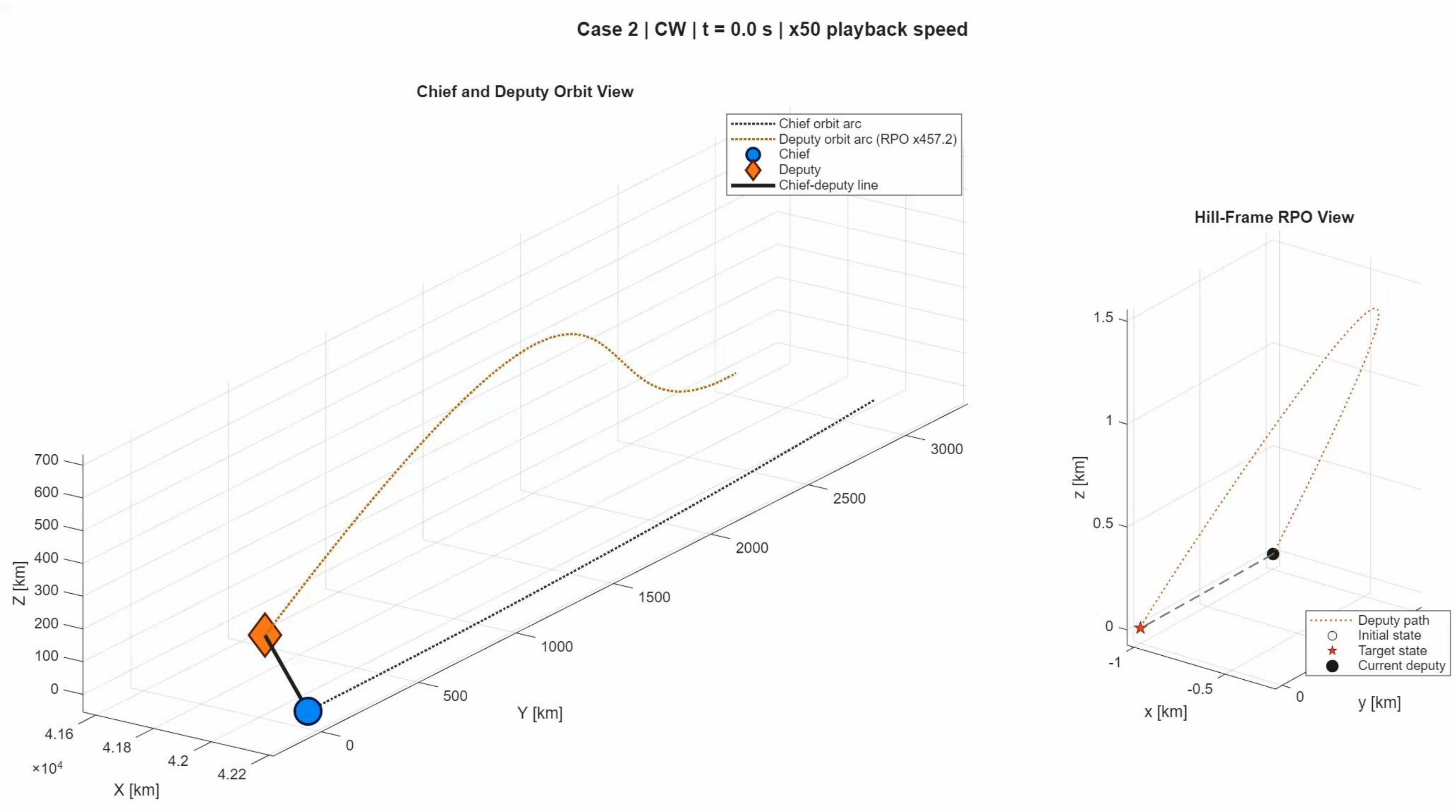
Case 2: Offset Terminal State

- Rendezvous from relative offset with more energetic initial velocity than Case 1, including strong in-plane and out-of-plane motion
- Target a prescribed offset final state rather than docking at the origin
- Completed over 1000 s with zero terminal error
- Peak control: 0.0451 m/s^2
- Control cost (integrated control effort): 0.2555 m^2/s^3



Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 2	-1.0	1.0	0.0	5.0	0.0	10.0	42169.0	0.0	0.0	3074.7	1000	-1.0	0.0	0.0	0.1	0.0	0.0

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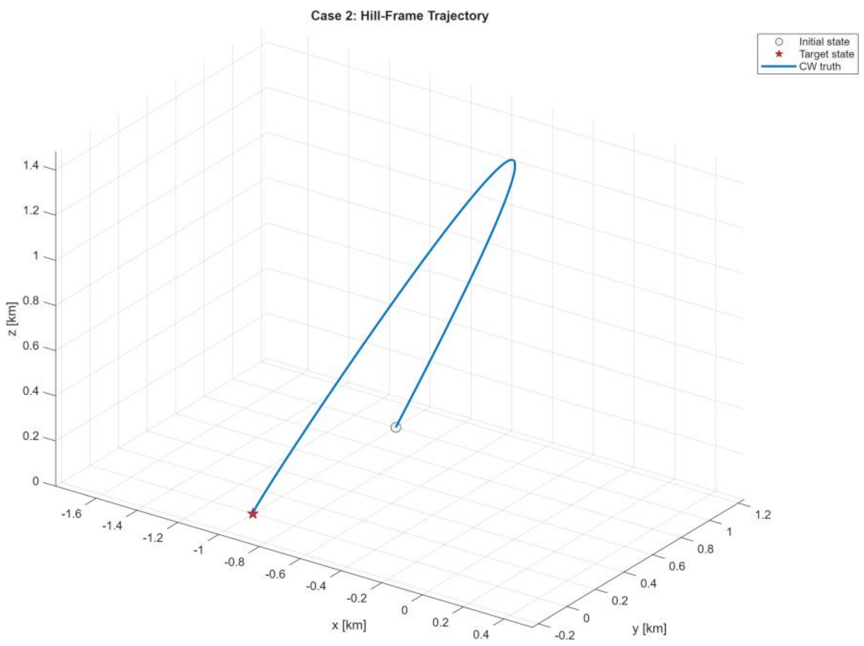
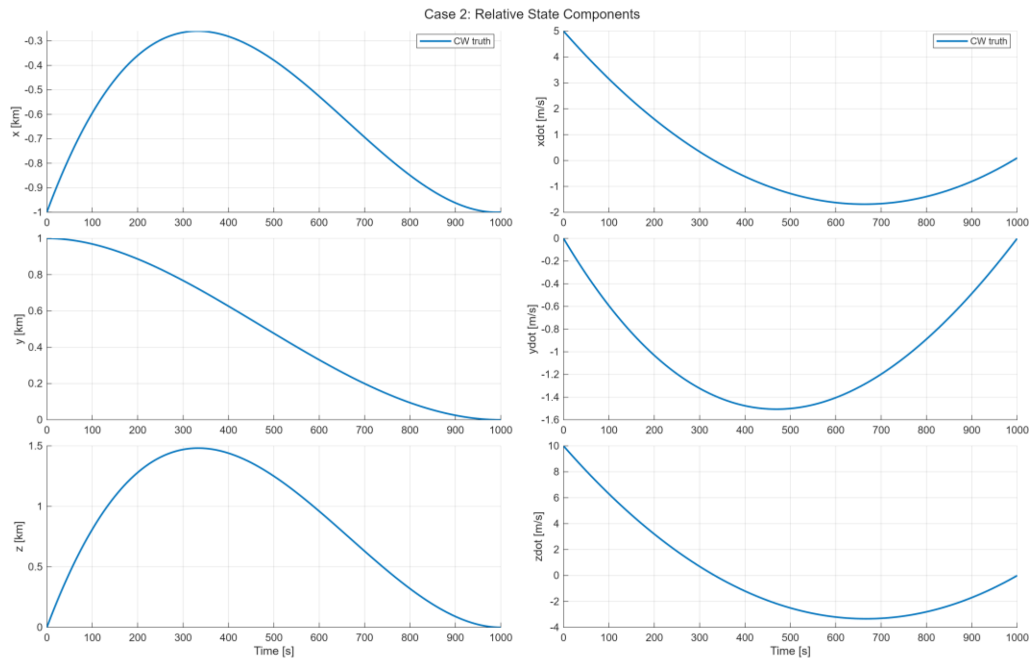
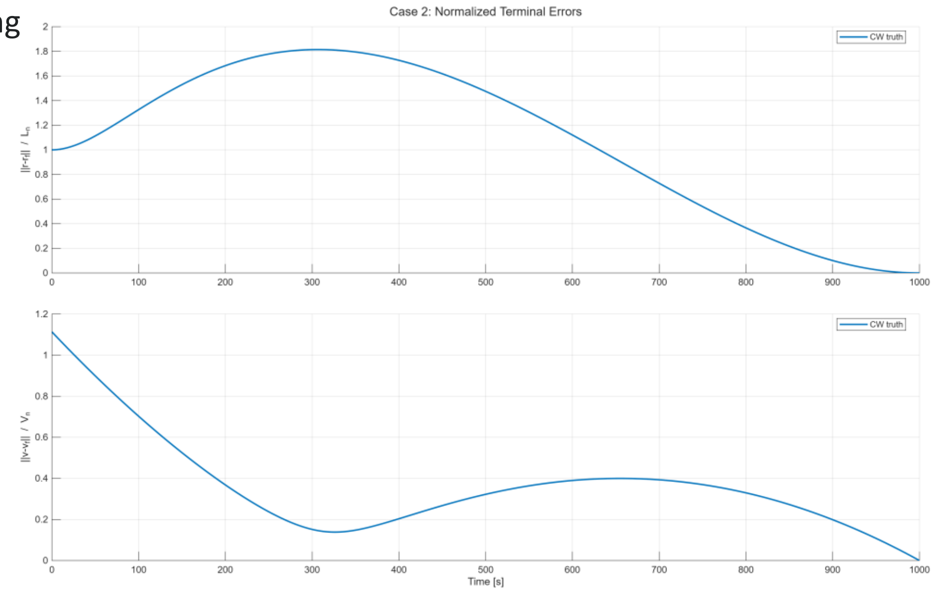
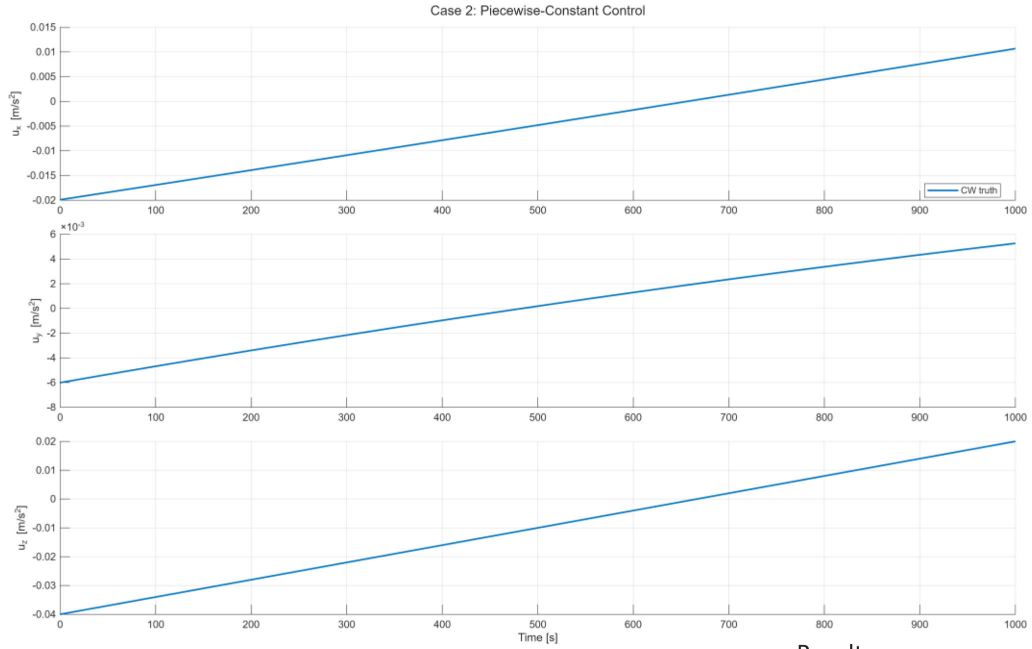




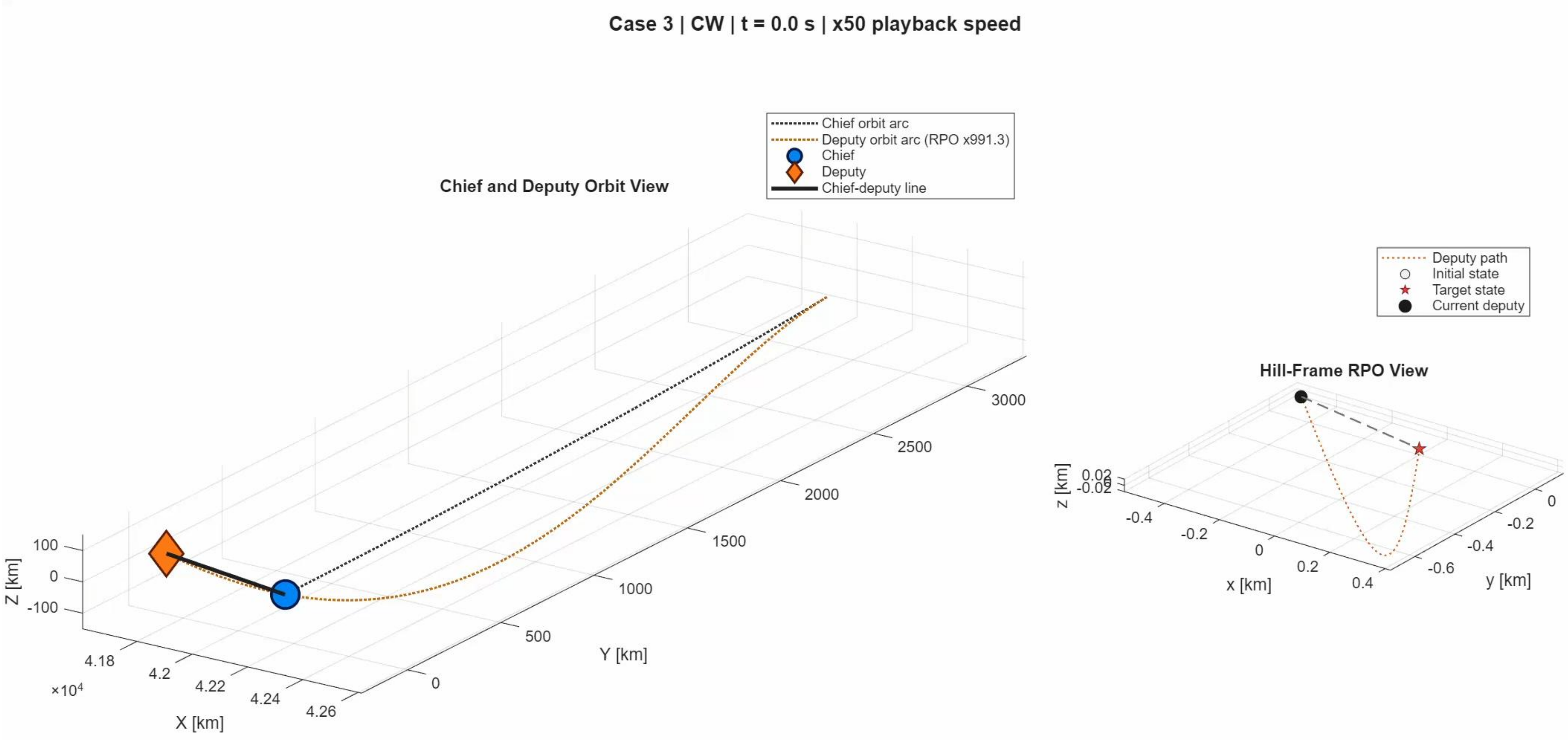




Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 2	-1.0	1.0	0.0	5.0	0.0	10.0	42169.0	0.0	0.0	3074.7	1000	-1.0	0.0	0.0	0.1	0.0	0.0

Case 3: Alternate 3D Docking Geometry

- Rendezvous from smaller initial offset than Cases 1 and 2, but with opposing in-plane velocity components that created a different geometrical approach
- Target full docking at the origin with zero terminal velocity
- Completed over 1000 s with zero terminal error
- Peak control: 0.0259 m/s^2
- Control cost (integrated control effort): 0.0841 m^2/s^3



Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 3	-0.5	0.1	0.0	5.0	-5.0	0.0	42169.0	0.0	0.0	3074.7	1000	0.0	0.0	0.0	0.0	0.0	0.0

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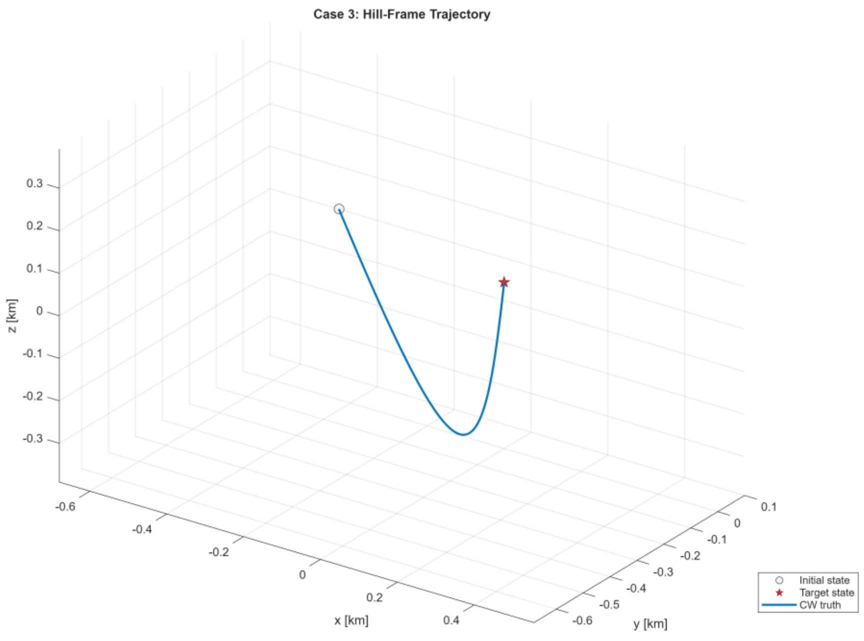
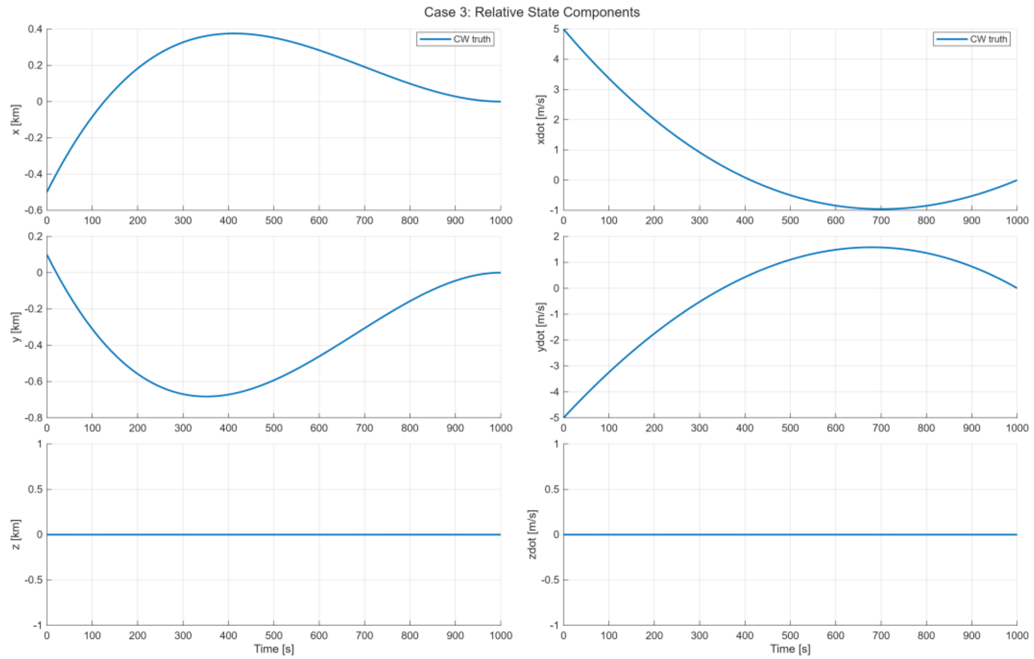
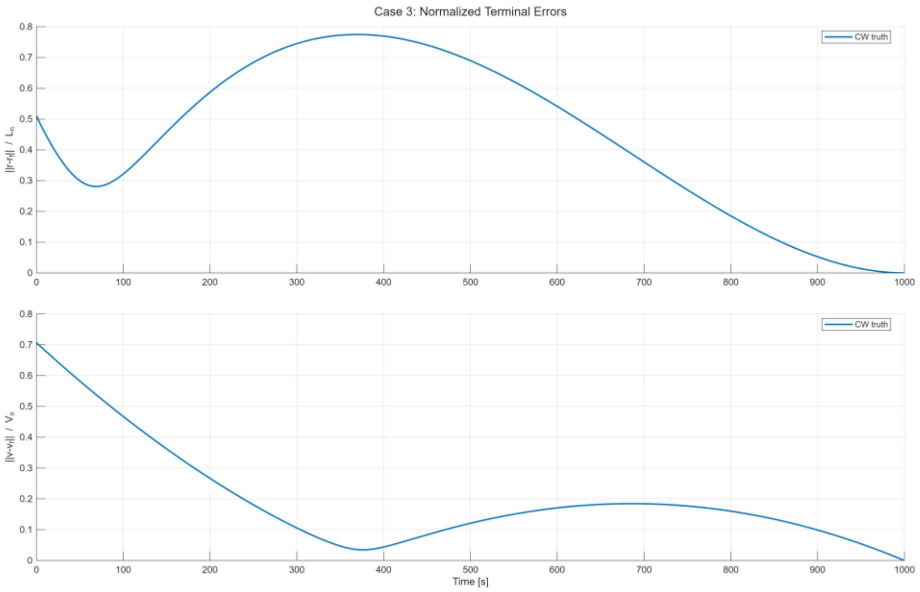
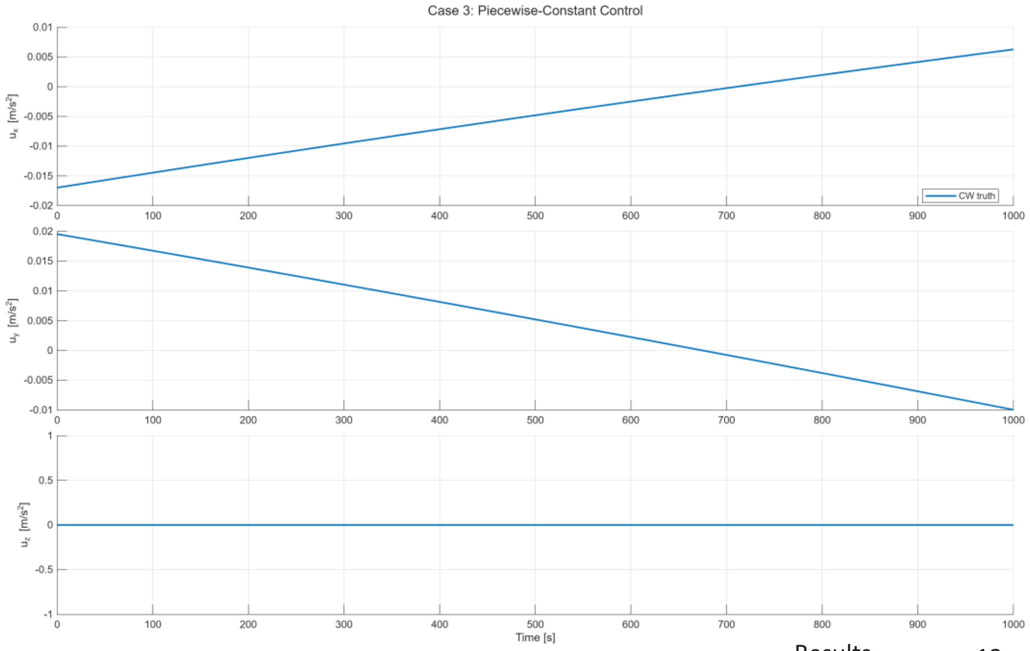




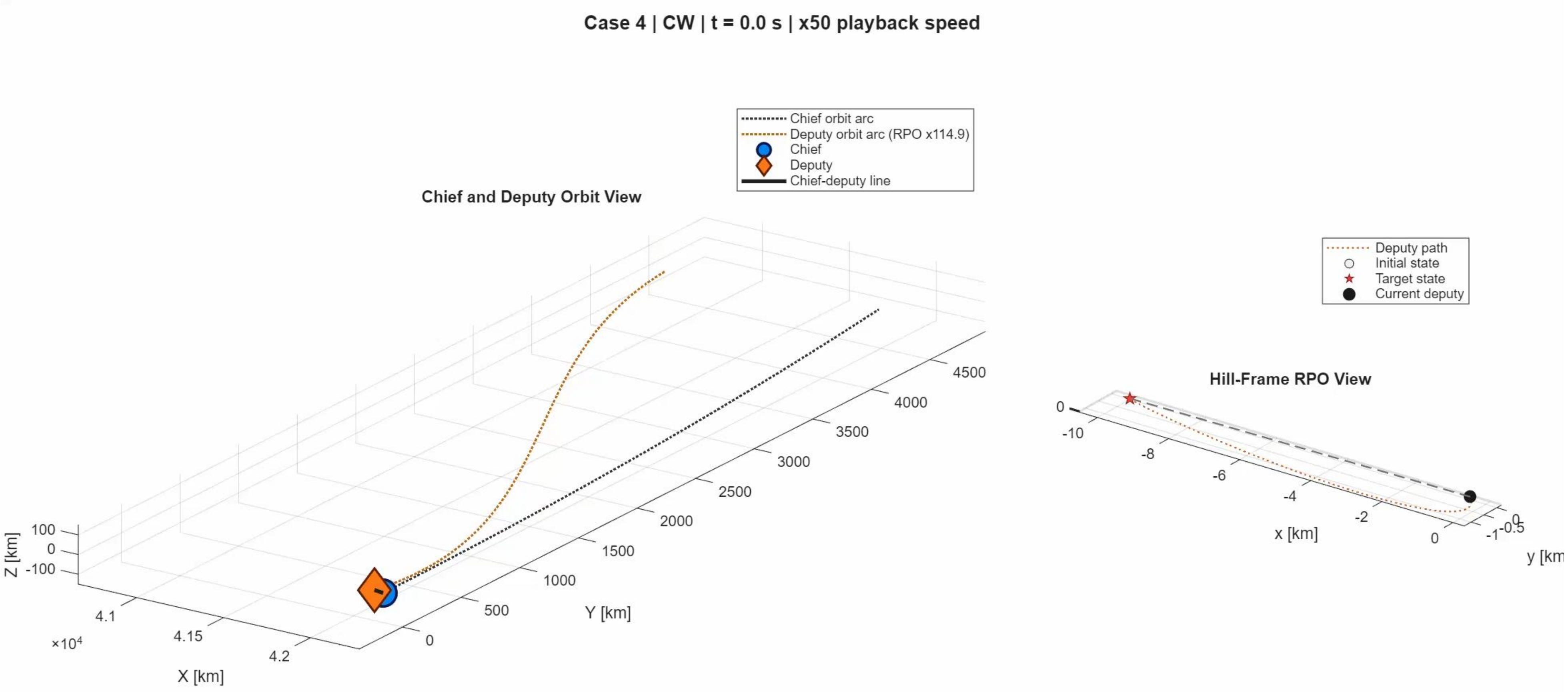




Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 3	-0.5	0.1	0.0	5.0	-5.0	0.0	42169.0	0.0	0.0	3074.7	1000	0.0	0.0	0.0	0.0	0.0	0.0

Case 4: Long-Horizon Offset Rendezvous

- Rendezvous from same initial condition as Case 3, but longer maneuver horizon to reach a different terminal state
- Target offset rendezvous with a prescribed final velocity instead of origin docking
- Completed over 1500 s with zero terminal error
- Peak control: 0.0415 m/s^2
- Control cost (integrated control effort): 0.3771 m^2/s^3



Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 4	-0.5	0.1	0.0	5.0	-5.0	0.0	42169.0	0.0	0.0	3074.7	1500	-10.0	0.0	0.0	1.0	0.0	0.0

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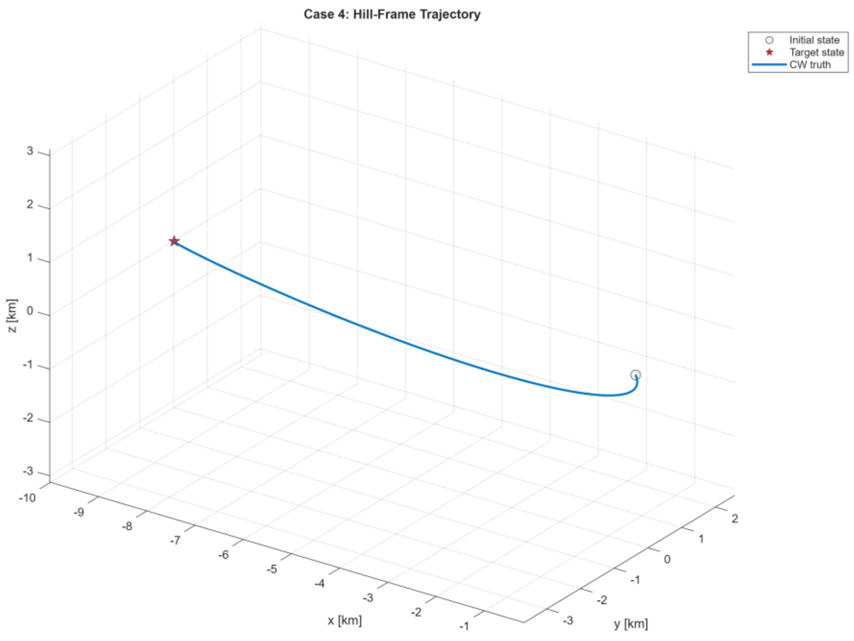
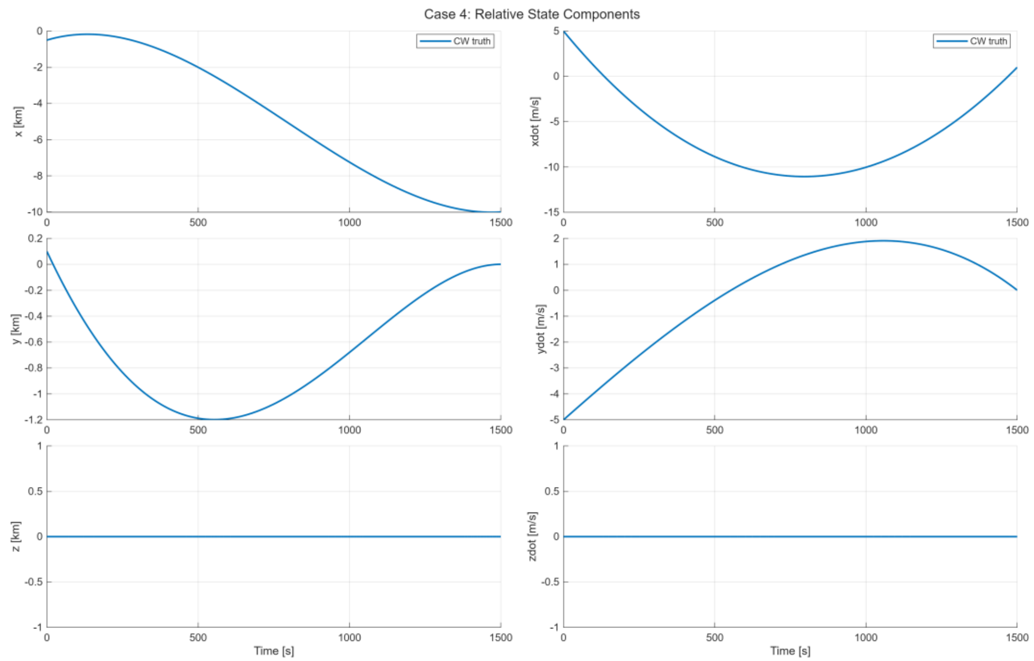
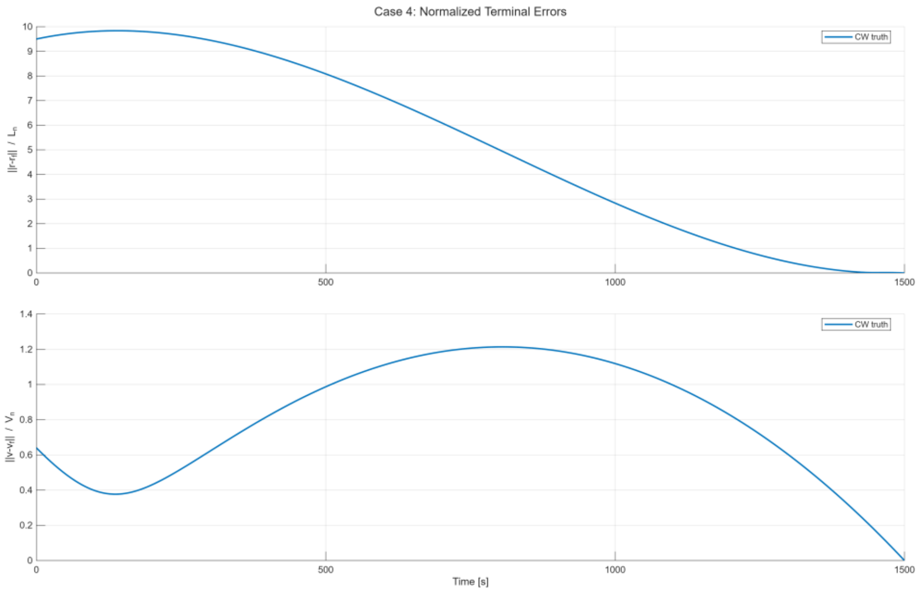
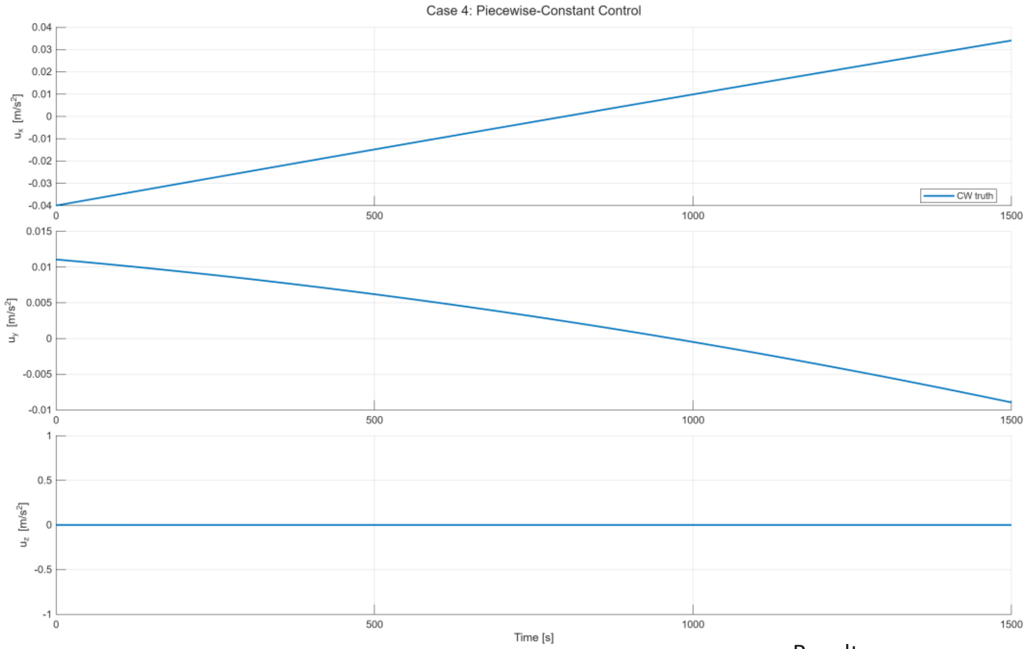




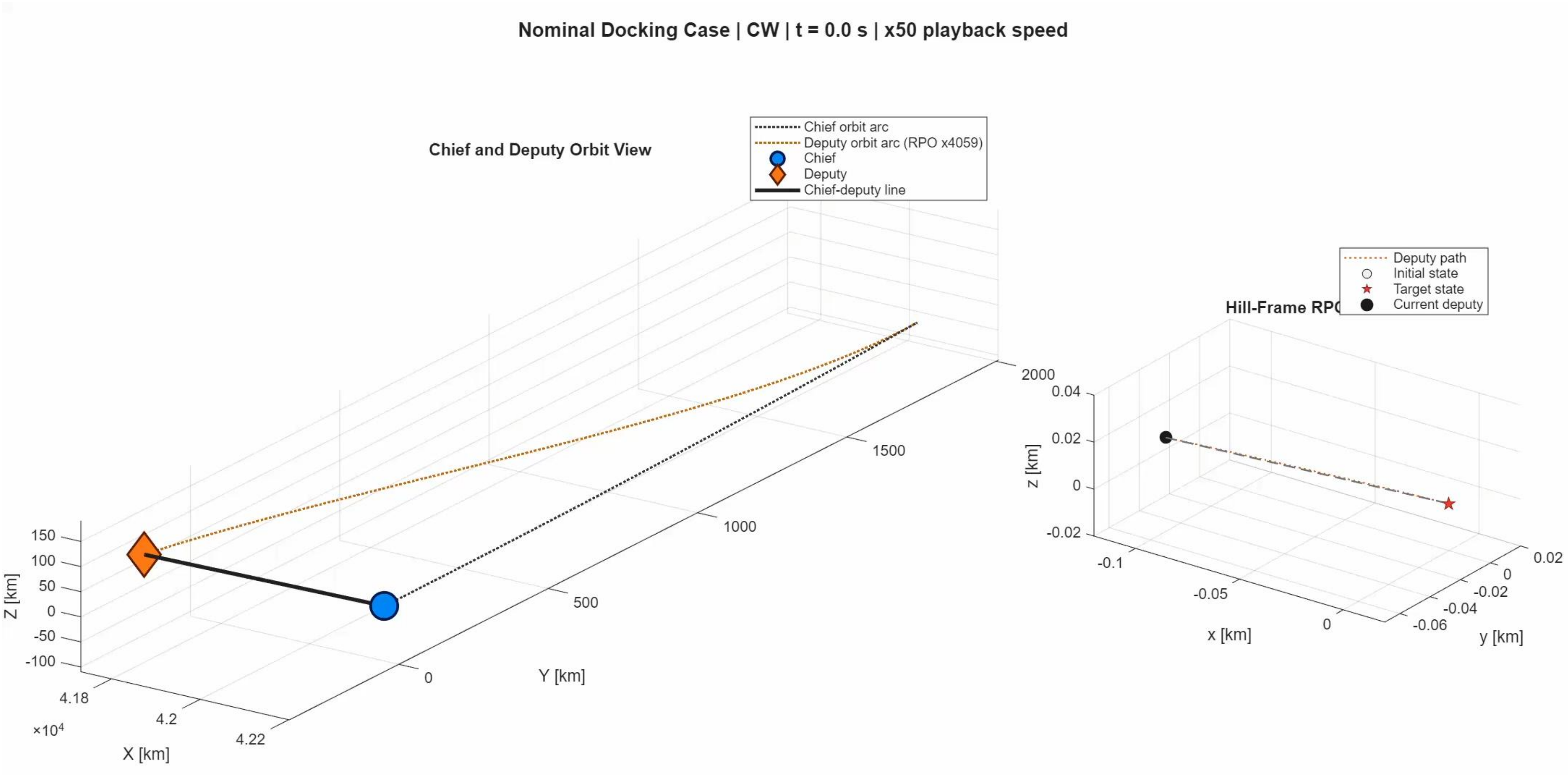




Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Case 4	-0.5	0.1	0.0	5.0	-5.0	0.0	42169.0	0.0	0.0	3074.7	1500	-10.0	0.0	0.0	1.0	0.0	0.0

Nominal Docking (International Space Station)

- Rendezvous from small offset with mild initial closing motion near the target
- Targeted gentle docking at the origin with zero terminal velocity
- Completed over 600 s with zero terminal error
- Peak control: 0.00174 m/s^2
- Control cost (integrated control effort): 3.16e-4 m^2/s^3



Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Nominal Dock	-0.1	-0.05	0.02	0.0	0.2	-0.1	42169.0	0.0	0.0	3074.7	600	0.0	0.0	0.0	0.0	0.0	0.0

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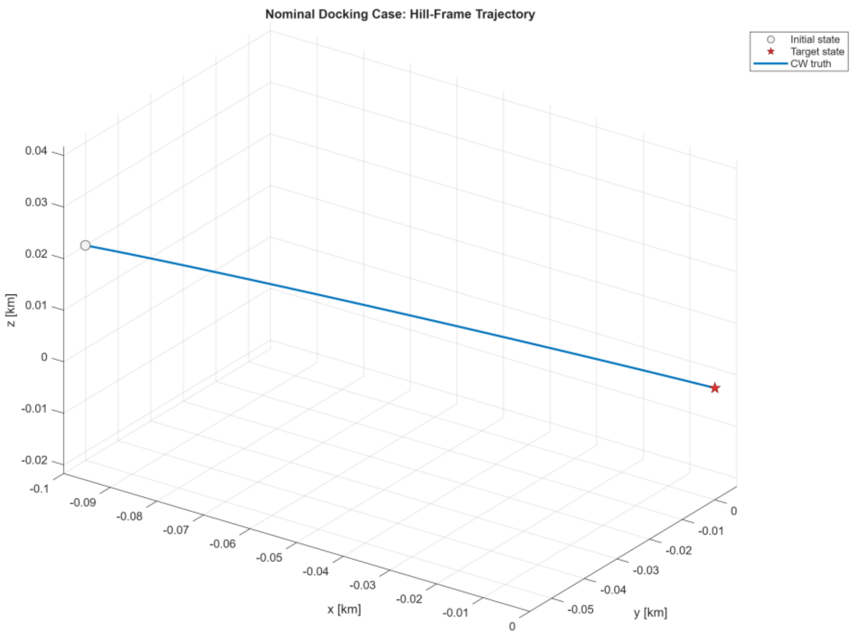

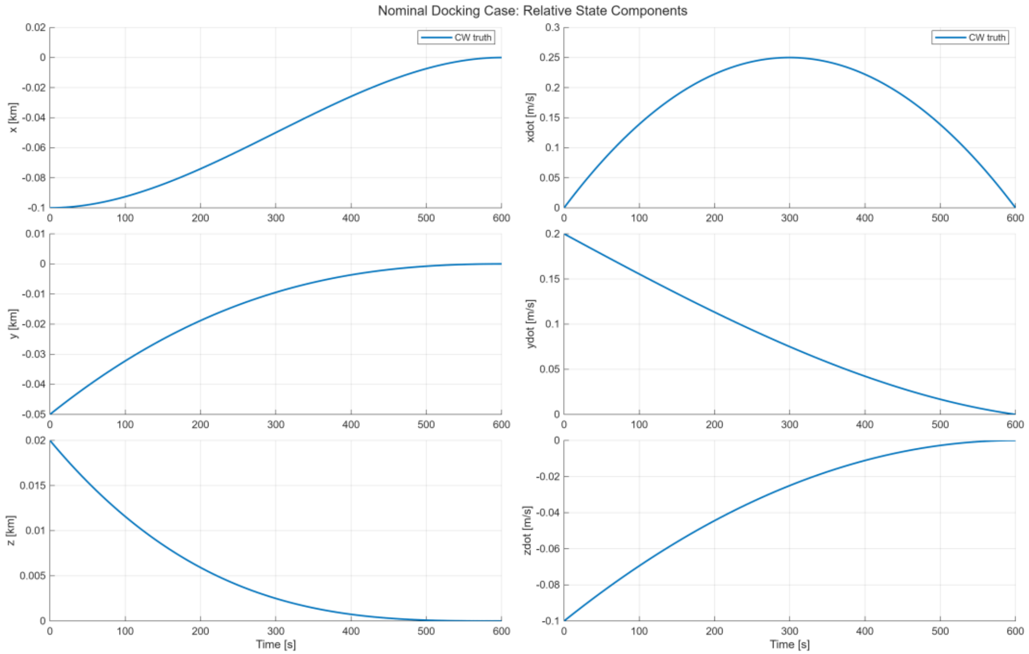

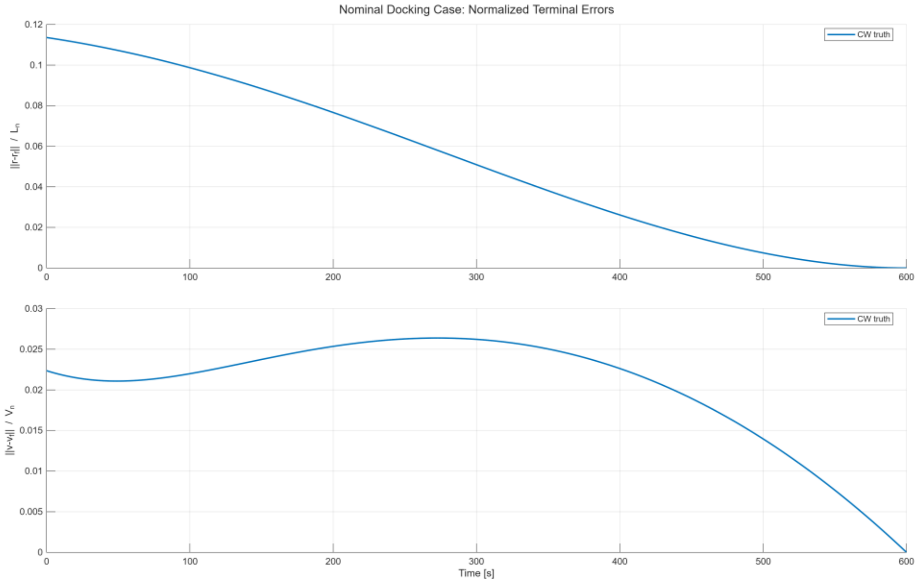

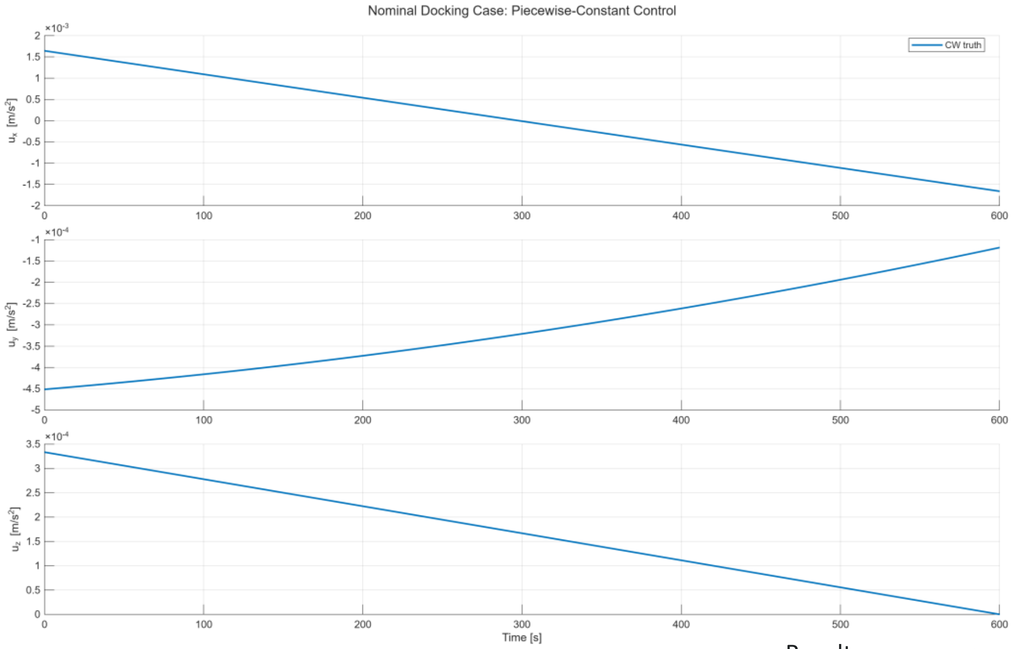

Case	x_0 (km)	y_0 (km)	z_0 (km)	$v_{x,0}$ (m/s)	$v_{y,0}$ (m/s)	$v_{z,0}$ (m/s)	$r_{T,x}$ (km)	$r_{T,y}$ (km)	$v_{T,x}$ (m/s)	$v_{T,y}$ (m/s)	t_f (s)	x_f (km)	y_f (km)	z_f (km)	$v_{x,f}$ (m/s)	$v_{y,f}$ (m/s)	$v_{z,f}$ (m/s)
Nominal Dock	-0.1	-0.05	0.02	0.0	0.2	-0.1	42169.0	0.0	0.0	3074.7	600	0.0	0.0	0.0	0.0	0.0	0.0

Summary and Future Work

Generalizations and Assumptions

- Circular chief orbit yields an LTI system (constant A and B matrices)
- Deputy and chief modeled as point masses with small relative separation (CW valid)
- Continuous idealized thrust actuation assumed
- Full state knowledge with no disturbances (J2, SRP, sensor noise neglected)
- No collision avoidance, keep-out zones, or approach corridor constraints

Key Takeaways

- CW equations provide a linear state-space model enabling real-time relative orbit guidance
- LQR-based minimum-energy law achieves near zero terminal error across all five test scenarios (in this idealized case)
- Control cost scales with initial energy and maneuver complexity
- Guidance law generalizes across diverse scenarios: docking, offset terminal states, and long-horizon maneuvers

Future Work

- Extend to nonlinear MPC for higher-fidelity dynamics
- Multi-spacecraft formation flight coordination
- Incorporate orbital perturbations (J2, SRP) and sensor noise
- Add collision avoidance and approach corridor constraints
- Hardware-in-the-loop validation and flight testing

Thank you!

Questions?

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More Information on this Project:

https://maxheil5.github.io/

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